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A note on the stability of an infinite fluid heated from below

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The problem of the stability of a fluid with time-dependent heating has been investigated by Morton (1957), Lick (1965) and Foster (1965). Morton and Lick assumed that the rate of change of the temperature profile is small compared with the growth rate of the disturbances (quasi-static assumption). This assumption is invalid near the onset of instability (as defined by $\partial/\partial t = 0$), and Foster has therefore used an initial-value approach.

In this paper the range of validity of the quasi-static assumption is discussed, and results of a time-scaled analysis and calculations based on this are compared with the work of Foster; the agreement is found to be good. We restrict our attention to a semi-infinite fluid initially at a constant temperature; at time $t = 0$ a temperature difference ΔT is applied at the (lower) horizontal boundary (case (A) of Foster).

The equations of the Boussinesq approximation are

$$\frac{\partial}{\partial t}\nabla^2 w = -g\alpha\,\nabla_H^2\theta + \nu\,\nabla^4 w, \tag{1}$$

$$\frac{\partial\theta}{\partial t} = -\frac{\partial T_0(z,t)}{\partial z}\,w + \nabla^2\theta, \tag{2}$$

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)w = -g\alpha\frac{\partial T_0(z,t)}{\partial z}\nabla_H^2 w, \tag{3}$$

where w is the vertical perturbation velocity, θ is the perturbation temperature, $T_0 = T_{00} + \Delta T\,\mathrm{erfc}\,(\frac{1}{2}z/(\kappa t)^{\frac{1}{2}})$ and

$$\nabla_H^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

In an infinite region possible length scales are $(\kappa t)^{\frac{1}{2}}$, $A^{-\frac{1}{3}}$ $(A = g\alpha\Delta T/\kappa\nu)$. The cube of the ratio of these two length scales is the time-scaled Rayleigh number

$$R_t = A(\kappa t)^{\frac{3}{2}} = \frac{g\alpha\,\Delta T\,(\kappa t)^{\frac{3}{2}}}{\kappa\nu}.$$

Scaling with a length scale of $(\kappa t)^{\frac{1}{2}}$ appropriate to the basic temperature profile(equivalent to a change of independent variables from $t, \mathbf{x}$ to $t, \mathbf{x}' = \mathbf{x}/(\kappa t)^{\frac{1}{2}}$)

and a temperature difference ΔT, i.e. $t = t'$, $x = (\kappa t)^{\frac{1}{2}} x'$, $w = (\kappa/t)^{\frac{1}{2}} w'$, $\theta = \Delta T\, \theta'$, gives (dropping the primes)

$$\left(\nabla^2 + \frac{1}{\sigma}\left[-t\frac{\partial}{\partial t} + \frac{3}{2} + \frac{x}{2}\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + \frac{z}{2}\frac{\partial}{\partial z}\right]\right)\nabla^2 w = R_t \nabla_H^2 \theta, \tag{4}$$

$$\left(\nabla^2 - t\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + \frac{z}{2}\frac{\partial}{\partial z}\right)\theta = -\frac{1}{\sqrt{\pi}}\exp\left(\frac{-z^2}{4}\right) w, \tag{5}$$

where σ is the Prandtl number, $\sigma = \nu/\kappa$.

It is usual when considering a problem with a time-independent basic state to define the onset of instability by the criterion $\partial/\partial t = 0$, noting that for a very small rate of growth the original perturbation will eventually reach a sizeable magnitude and that the time taken for this growth is of no importance in the definition of such a state as stable or unstable. However, in the present problem we have a time-dependent basic state and may therefore choose to define the onset of instability as occurring at that time when the perturbation becomes large enough for non-linear effects to be important; or (we assume, equivalently) when the disturbance is first physically detectable. It is this time (or equivalently $R_t = A(\kappa t)^{\frac{3}{2}}$, when any initial perturbation has grown by several orders of magnitude and is growing superexponentially, that we wish to calculate.

If the assumption is made that the rate of growth is of order unity while R_t is of order unity (as seems reasonable from examination of (4), (5); and as demonstrated by the work of Foster), the perturbation will still be infinitesimally small when R_t reaches a large value and the linear theory remains valid at this time.

When R_t is large we may carry out a two-time analysis of the problem, introducing the time scales

$$t_* = t(1 + R_t^{-a} w_1 + \ldots),$$

$$t_1 = R_t^b t,$$

and expanding the velocity and temperature asymptotically:

$$\theta = \theta(t_*, t) + \ldots,$$

$$W = R_t^c W(t_*, t_1) + \ldots.$$

Thus $t(\partial/\partial t) \approx R_t^b t_*(\partial/\partial t_1) \sim R_t^b$ and b, c are chosen so that both sides of (4), (5) are of equal orders of magnitude.

This analysis is equivalent to that following from the quasi-static assumption (that the rate of change of the basic temperature profile is small compared to the growth rate of the disturbance, i.e. $t(\partial/\partial t) \gg 1$) and makes clear the region of validity of that assumption. Since the quasi-static assumption does not hold at the onset of instability as defined by $t(\partial/\partial t) = 0$, the Rayleigh number noted by Lick (1965) for this has no significance, the theory used being invalid at that time. (The time-scaled Rayleigh number quoted in that paper for this onset of instability, $R_t \approx 300$, is a misprint, the value obtained from figure 5 being $R_t \approx 5{\cdot}4$.)

(i) *Large Prandtl number* $1/\sigma \to 0$

A two-time analysis of (4) and (5) yields the following first-order equations for large R_t:

$$\nabla^4 w' = -a_t^2 \theta, \tag{6}$$

$$n\theta = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{z^2}{4}\right) w', \tag{7}$$

where $t\,\partial/\partial t = nR_t$, $w' = R_t^{-1} w$, a_t is the scaled horizontal wave-number (provided the region is effectively infinite, i.e. $(\kappa t)^{\frac{1}{2}} \ll L$, the relevant length scale is $(\kappa t)^{\frac{1}{2}}$ and the physical wavelength varies with time).

The solution is obtained by expanding the temperature perturbation in a Fourier sine series and solving the resulting set of coupled differential equations. This method is outlined in Chandrasekhar (1961, p. 53).

For the maximum rate of growth we have

$$t\frac{\partial w}{\partial t} = n_{\max} R_t w$$

and from this we obtain

$$\log_{10} w = \log_{10} w_0 + 0{\cdot}434\,\tfrac{2}{3} n_{\max} R_t. \tag{8}$$

For free horizontal boundary conditions (zero vertical velocity, zero tangential stress) the rate of growth is a maximum at a wave-number $a_t \approx 0{\cdot}48$ and the velocity then obeys the equation

$$\log_{10} w = \log_{10} w_0 + 0{\cdot}039 R_t.$$

For rigid horizontal boundary conditions (zero vertical and horizontal velocities) the rate of growth is a maximum at a wave-number $a_t \approx 0{\cdot}9$ and the velocity then obeys the equation

$$\log_{10} w = \log_{10} w_0 + 0{\cdot}02 R_t.$$

The wave-numbers for maximum growth, the above dependence of w on R_t, and the coefficients agree with data supplied by Foster. Some of these data are found in Foster's paper; other data including result up to $w = 10^8$ were obtained by private communication. In both cases the data are fitted by choosing

$$\log_{10} w_0 = -0{\cdot}37, \quad \text{i.e.} \quad w_0 = 0{\cdot}43.$$

Thus the initial infinitesimal perturbation changes in magnitude by a factor of 0·43 before the quasi-static assumption becomes valid.

The Prandtl number is considered large when $(1/\sigma)\,t\,\partial/\partial t \ll 1$ and, since $t\,\delta/\partial t \sim R_t$ here, this condition is $\sigma \gg R_t$.

(ii) *Small Prandtl number*

A two-time analysis of (4), (5) yields in this case the following first-order equations for large R_t:

$$-n\nabla^2 w'' = -a_t^2 \theta, \tag{9}$$

$$n\theta = \frac{1}{\sqrt{\pi}} \exp\left(\frac{-z^2}{4}\right) w'', \tag{10}$$

where $t\,\partial/\partial t = (\sigma R_t)^{\frac{1}{2}} n$, $w'' = (\sigma R_t)^{-\frac{1}{2}} w$, a_t is the scaled horizontal wave-number. The eigenvalue problem is now of lower order and the boundary conditions are that the vertical velocity is zero on the horizontal boundary and at $z = \infty$.

For the maximum rate of growth we have $t\,\partial w/\partial t = n_{\max}(\sigma R_t)^{\frac{1}{2}} w$ and from this we obtain

$$\log_{10} w = \log_{10} w_0 + 0{\cdot}434 \frac{4}{3} n_{\max}(\sigma R_t)^{\frac{1}{2}}. \tag{11}$$

The rate of growth is a maximum at a wave-number of $a_t \sim 4{\cdot}5$–$5{\cdot}0$ and the velocity then obeys the equation

$$\log_{10} w = \log_{10} w_0 + 0{\cdot}38(\sigma R_t)^{\frac{1}{2}}.$$

Foster's calculations show $a_t \sim 4$ and his equation for the velocity is

$$\log_{10} w = \log_{10} w_0 + 0{\cdot}22(\sigma R_t)^{\frac{1}{2}}.$$

This relation was calculated from data supplied by Foster in a private communication. It is interesting to note that the second and third eigenvalues for $n_{\max}$ in the present work give the velocity equations

$$\log_{10} w = \log_{10} w_0 + 0{\cdot}29(\sigma R_t)^{\frac{1}{2}} \quad \text{and} \quad \log_{10} w = \log_{10} w_0 + 0{\cdot}19\,(\sigma R_t)^{\frac{1}{2}}.$$

The Prandtl number is considered small when $\sigma^{-1} t\delta\, \partial/\partial t \gg 1$ and since

$$t\,\partial/\partial t \sim (\sigma R_t)^{\frac{1}{2}}$$

here, this condition is $\sigma \ll R_t$.

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